Angle Modulations

- With AM, we use the envelope (amplitude) of the carrier to convey the data that is to be sent to the receiver.

- Instead of using the amplitude, we can use the phase (angle) of the carrier to convey info. The resulting techniques are known as angle modulations.

- The angle modulated signal can be expressed as
  \[ y(t) = A \cos \left( \omega_c t + \theta(t) \right), \]
  where \( \theta(t) \) is the phase function which is produced from the info signal \( m(t) \).

- We define the instantaneous freq
  \[ \dot{\omega}_i(t) = \frac{d\theta(t)}{dt}, \]
  so that
  \[ \theta(t) = \int \dot{\omega}_i(t) \, dt. \]
  An analogy to the relationship between \( \dot{\omega}_i(t) \) & \( \theta(t) \) is the relationship between velocity and displacement in elementary physics.

- There are two common ways to obtain \( \theta(t) \) from \( m(t) \):

  (i) Phase Modulation (PM)
  \[ \theta_p(t) = k_p m(t), \quad \text{i.e.} \quad y(t) = A \cos(\omega_c t + k_p m(t)) \]
  \[ \dot{\omega}_i = \omega_c + k_p \frac{dm(t)}{dt} \]

  (ii) Frequency Modulation (FM)
  \[ \theta_f(t) = k_f \int_{-\infty}^{t} m(x) \, dx, \quad \text{i.e.} \quad y(t) = A \cos(\omega_c t + k_f \int_{-\infty}^{t} m(x) \, dx) \]
  \[ \dot{\omega}_i(t) = \omega_c + k_f m(t) \]
One can generalize the idea of PM to FM to obtain a generalized angle-modulation:

$$\theta(t) = \int_{-\infty}^{t} M(\tau) R(t-\tau) d\tau$$

when $R(t) = R_p S(t)$, we have PM.

when $R(t) = R_c \cos(\omega_c t)$, we have FM.

Examples: See Examples 5.1 and 5.2 in text.

Power of angle-modulated signal:

when $E = \max |W_i(t)|$, we see from the examples above that we are integrating over multiple periods of the carrier, thus the power is simply $\frac{A^2}{2}$.

Bandwidth of angle-modulated signal

First notice that the BW is not $\max W_i(t) - \min W_i(t)$.

To see this, write

$$y(t) = A \cos(\omega_c t + \theta(t)) = Re \left[ A e^{j\omega_c t} e^{j\theta(t)} \right]$$

$$= Re \left[ A e^{j\omega_c t} \sum_{k=0}^{\infty} \frac{\theta_n}{n!} t^n \right]$$

$$= A \left[ \cos(\omega_c t) + \frac{\theta_1}{1!} \sin(\omega_c t) \cos(\omega_c t) \right.$$

Notice that $\theta_n(t)$ for a BW that is $n$ times that of $\theta_0(t)$. (Why?) Thus $y(t)$ is not bandlimited and the essential BW may be very large.
To obtain the BW of an angle-modulated signal in general, instead we will get a simple result which can serve as a design rule-of-thumb design, we once again specialize to the case of FM modulation, i.e., the message is a sinusoidal signal:

\[ m(t) = \alpha \cos(\omega t) \quad \text{and} \quad \int_{-\infty}^{\infty} |m(t)|^2 \, dt = \frac{\alpha^2}{W_m} \sin(m \omega t) \]

Then for PM, \( \Theta(t) = \omega t + \beta m(t) = \omega t + \beta \alpha \cos(\omega t) \)

For FM, \( \Theta(t) = \omega t + \beta \int_{-\infty}^{\infty} m(t) \cos(m \omega t) \, dt = \omega t + \frac{\beta \alpha}{W_m} \sin(m \omega t) \)

First, let's consider FM, \( y(t) = A \cos[\alpha e^{j\Theta(t)}] \)

Let's define freq deviation \( \Delta \omega = \max_{\omega > \omega_c} \omega - \min_{\omega < \omega_c} \omega \).

In this case, \( \dot{\Theta}(t) = \frac{d\Theta(t)}{dt} = \omega t + \beta \alpha \cos(\omega t) \)

Thus, \( \Delta \omega = \beta \alpha \).

Notice that the BW of \( m(t); B = \omega_m / 2\pi = \text{fm} \) \( \text{Hz} \)

We define deviation ratio \( \beta = \frac{\Delta \omega}{2\pi B} \)

Thus \( \beta = \frac{\beta \alpha}{W_m} \) in the case of FM and same modulator

Rewrite \( y(t) \) in terms of \( \dot{\Theta} \) & \( \Theta \):

\[ y(t) = A \cos[\alpha e^{j\theta(t)}] \]

Notice that \( e^{j\theta(t)} \) is periodic for \( \omega_m \) with period \( 2\pi \), it has a Fourier series expansion:

\[ e^{j\theta(t)} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn \omega t} \]
where \( J_n(\beta) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - \beta x)} dx \) is the \( n \)-th order Bessel function of the first kind, which is well-tabulated and available in, e.g., MATLAB.

From the plot of \( J_n(\beta) \) above, we see that \( J_n(\beta) \) is small when \( n > \beta + 1 \).

Plugging the Fourier series expansion of \( e^{j\beta \sin t} \) back to the expression for \( y(t) \), we get:

\[
y(t) = \text{Re} \left[ A \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j[\omega t + n\omega_M t]} \right] = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega t + n\omega_M t)
\]

Thus, we see that \( y(t) \) is a sum of sinusoids at \( \omega + n\omega_M \), each scaled by \( J_n(\beta) \). Since \( J_n(\beta) \) is small when \( n > \beta + 1 \), only those sinusoids with \( \beta \) closest to \( \omega + n\omega_M \) for \( n = 0, 1, \ldots, \beta + 1 \) are significant. So the BW of the FM signal \( y(t) \) can be obtained roughly as:

\[
2(\beta + 1) \Delta \omega = 2(\Delta \omega + 2\beta B) > 2\beta B \quad (\text{BW of AM signal})
\]
For PM, the situation is similar:
\[ \Delta w = \frac{k_p}{X_m}, \quad \beta = \frac{k_p}{X_m} \]
\[ y(t) = \text{Re} \left[ A e^{j \omega t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j n \omega t + \phi} \right] \]

Notice that \[ e^{j n \omega t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j n \omega t + \phi} \]

Plugging this back into \( y(t) \), we have, like before,
\[ y(t) = A \sum_{n=0}^{\infty} J_n(\beta) \cos(\omega t + n \omega t + \phi) \]

As before, \( J_n(\beta) \) is small when \( n > \beta + 1 \) and
the BW of the PM signal is roughly
\[ 2(\beta + 1)X_m = 2(\omega m + 2\beta X_m) > 4\pi X_m \]

Example: See text Example 5.3.

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**Immunity to nonlinear distortions**

Angle-modulated signals are much more immune to nonlinear distortions caused by the nonlinearities in the transmitter (usually power amplifiers). To see this, let us assume that an angle-modulated signal \( y(t) = A \cos(\omega t + \theta_0(t)) \) is distorted by a power amplifier which gives output \( x(t) = ay(t) + b y^3(t) \).

Thus
\[ x(t) = a A \cos(\omega t + \theta_0(t)) + b A^3 \cos^3(\omega t + \theta_0(t)) \]

\[ = (a A + \frac{3b}{4} A^3) \cos(\omega t + \theta_0(t)) + \frac{b}{4} A^3 \cos^3(\omega t + \theta_0(t)) \]
We can see the nonlinearity causes an additional triple-frequency term at the output. This is not too much a problem since we can easily remove this triple-frequency term by filtering.

This behavior extends to the general case of nonlinearities that can be described by a polynomial. The nonlinear distortions are harmonic terms up to the nth harmonic where $n$ is the order of the polynomial. (Note that this property can be used to implement a freq. multiplication circuitry. Can you figure how?)

On the other hand, AM signals do not have the above immunity against nonlinear distortion. Suppose that a DSB-SC signal $y(t) = A\cos(\omega t)$ is passed through the same power amplifier; the output

$$x(t) = ay(t) + b y^3(t) = aA\cos(\omega t) + bA^3\cos^3(\omega t)$$

$$= (aA\cos(\omega t) + \frac{3ba^2}{4} \cos^3(\omega t)) + \frac{b}{4} a^3 \cos(\omega t).$$

Notice that filtering alone can only remove the triple-frequency term in this case, the resulting signal still contains the distorted version of the message in the amplitude of the carrier.

Because of this vulnerability against nonlinear distortions, FM (PM) is used in applications that it is expensive to build linear amplifiers.
Generation of FM (PM).

(i) **Direct Method**
- Instantaneous freq of FM signal determined by message:
  \[ \omega_a(t) = \omega_c + k t m(t) \]
  
So can use a VCO with input \( m(t) \) to generate an FM signal.

- Need to stabilize the VCO by feeding back the loop-filtered error signal between VCO output & a stable freq reference (like a PLL).

(ii) **Indirect Method**
- Idea: First generate a narrowband FM (\( \beta < 1 \)) signal and then freq multiply the narrowband FM to a wide band FM (to desired value of \( \beta \)).
  
- The narrowband FM signal can be generated by the following circuit:

![Diagram of narrowband FM generator circuit]
To see why it works, let us recall that an FM signal
\[ y(t) = A \cos(\omega t + \Theta(t)) \quad (\Theta(t) = \int_0^t \omega(t') \, dt') \]
\[ = A [\cos \omega t - \Theta(t) \sin \omega t - \frac{\Theta''(t)}{2!} \cos 2 \omega t + \ldots] \]

If \( |\Theta(t)| \ll 1 \), then only the 1st 2 terms are significant, i.e.
\[ y(t) \approx A [\cos \omega t - \Theta(t) \sin \omega t] \]

We claim that \( |\Theta(t)| \ll 1 \) when \( \beta \ll 1 \).

To see this, let us consider the simple case of time
modulation in which \( \Theta(t) = \frac{\beta \alpha}{\omega_m} \sin \omega t \).

So \( \max |\Theta(t)| = \frac{\beta \alpha}{\omega_m} \), but \( \beta = \frac{\Delta \omega}{2 \pi \beta} = \frac{\Delta \omega}{\omega_m} = \frac{\omega_0}{\omega_m} \).

Thus \( \beta \ll 1 \Rightarrow \frac{\beta \alpha}{\omega_m} \ll 1 \Rightarrow |\Theta(t)| \ll 1 \).

As a result, for narrowband FM, we can approximately
generate \[ y(t) \approx A [\cos \omega t - \beta \int_0^t \omega(t') \, dt' \sin \omega t] \]
by a factor \( n \).

- The second step of freq. multiplication increases \( \beta \) and \( \alpha \)
  by the same factor \( n \).

- Example: Commercial FM transmitter on text p. 229 - 230
Demodulation of FM

(i) Frequency discriminators

Consider an FM signal \( y(t) = A(t) \cos(\omega_c t + B(t) \int_{-\infty}^{t} m(u) du) \).

Differentiating it, we get
\[
\frac{dy(t)}{dt} = A(t) (\omega_c + B(t)) \sin(\omega_c t + B(t) \int_{-\infty}^{t} m(u) du).
\]

Note that \( \Delta \omega = B(t) \max|m(t)| \ll \omega_c \). Thus the envelope of the derivative acts like the envelope of an AM signal.

This observation suggests the following receiver for FM:

\[\begin{array}{ccc}
\text{FM signal} & \rightarrow & \frac{d}{dt} \\
& & \text{Envelope detector} \\
& & \rightarrow & A(t) m(t)
\end{array}\]

The approach is called frequency discrimination. The problem with this approach is that the differentiation operation is very sensitive to noise & variation in the amplitude of the FM signal. For example, if

\[
y(t) = A(t) \cos(\omega_c t + B(t) \int_{-\infty}^{t} m(u) du)
\]

\[
\frac{dy(t)}{dt} = -A(t) (\omega_c + B(t)) \sin(\omega_c t + B(t) \int_{-\infty}^{t} m(u) du) \\
+ \frac{dA(t)}{dt} \cos(\omega_c t + B(t) \int_{-\infty}^{t} m(u) du)
\]

The envelope of this derivative is given by

\[
\sqrt{A^2(t) (\omega_c + B(t) m(t))^2 + \left[ \frac{dA(t)}{dt} \right]^2}
\]

Thus we could suffer from distortion due to \( A(t) \).
To solve this problem, we can use the bandpass limiter as shown below:

\[ A(t) \cos(\omega t + \phi(t)) \rightarrow \text{Hard limiter} \rightarrow \frac{4}{\pi} \cos(\omega t + \phi(t)) \rightarrow \text{Bandpass filter} \]

The FM signal \( y(t) = A(t) \cos(\omega t + \phi(t)) \) is first passed through a hard-limiter with S/O characteristic

\[ v_o(t) = \begin{cases} 1 & \text{if } v_{in} > 0 \\ -1 & \text{if } v_{in} < 0 \end{cases} \]

with \( y(t) \) as input (assuming \( A(t) > 0 \)), \( v_o(t) \) is a periodic \( f_0 \) of \( \phi(t) \):

\[
\begin{align*}
\text{Thus } v_o(\phi) &= \frac{4}{\pi} \left( \cos \phi - \frac{1}{3} \cos 3\phi + \frac{1}{5} \cos 5\phi + \ldots \right) \\
\text{Hence } v_o(y(t)) &= \frac{4}{\pi} \left[ \cos(\omega t + \phi(t)) - \frac{1}{3} \cos(3\omega t + 3\phi(t)) + \frac{1}{5} \cos(5\omega t + 5\phi(t)) + \ldots \right] \\
\end{align*}
\]

Notice that the amplitude variation \( A(t) \) is removed by the limiter and the freq info is kept intact.
- Recall that a PLL can be used to track the changes in the carrier freq & phase. Hence, the same circuitry can also be used to track the instantaneous freq of the FM signal, provided that the time constant of the feedback loop is small enough.

- Revisit the PLL with FM signal as input

\[ F(t) = A \cos(\omega t + \theta_0(t)) \]

\[ Y(t) = X(t) \]

\[ X(t) = -B \sin(\omega t + \theta_0(t)) \]

As before, \( \frac{d\Theta(t)}{dt} = K \cdot E_0(t) \) (VCO operation).

\[ X(t) = -B \sin(\omega t + \theta_0(t)) + \frac{AB}{2} \sin(\theta_0(t) - \theta(t)) \]

Again, we assume the loop filter is a LPF and hence removes the double freq term above. Thus in Laplace domain:

\[ E_0(s) = H(s) \cdot \frac{AB}{2} \sin(\theta_0(s) - \theta(s)) \]

\[ \approx H(s) \frac{AB}{2} \left[ \Theta_0(s) - \Theta(s) \right] \quad \text{when} \quad \Theta_0(t) - \Theta(t) \]

But \( s \Theta(s) = C \cdot F(s) \) (VCO operation in Laplace domain)

\[ s \Theta(s) = H(s) \frac{AB}{2} \left[ \Theta_0(s) - \Theta(s) \right] \]

\( \Rightarrow \frac{\Theta(s)}{\Theta_0(s)} = \frac{K(s)}{K(s) + s} \quad \text{where} \quad K = \frac{ABC}{2} \)
This if \(|K H(s)| \gg 1\) for the passband of \(H(s)\), the condition that the (this condition corresponds to the loop time constant is small compared with the reciprocal of the BW of the FM signal), then

\[
\theta_m(s) \approx \theta_0(s) \quad \Rightarrow \quad \theta_e(t) = \theta_0(t)
\]

Hence

\[
\theta_e(t) = \frac{1}{c} \frac{d}{dt} \theta_0(t) = \frac{1}{c} \frac{d}{dt} \int_{-\infty}^{t} \phi_m(\tau) d\tau = \frac{1}{c} M(t)
\]

The error signal of the PLL (at the output of the loop filter) measures the message!

- Because of the feedback structure, the PLL receiver is more immune to noise & interference than the freq. discriminator

- PLLs see text section 5.6 for description of canonical FM broadcast systems.