1. (20%) Valid or Not
For each of the functions below, determine whether it is a valid cdf (pdf). If so, determine whether the corresponding random variable is discrete or continuous. If not, give a reason why the function is not a valid cdf (pdf). No credit will be given to an answer without the elaboration describe above.

(a) Is this a valid cdf?

Yes, discrete r.v.

(b) Is this a valid cdf?

No

CDF must be non-decreasing.
(c) Is \( f(x) = \begin{cases} \frac{x}{2T} \sin \frac{2\pi x}{T} & \text{for} \; 0 < x < T \\ 0 & \text{otherwise} \end{cases} \) a valid pdf?

Yes, continuous r.v.

(d) Is \( f(x) = \begin{cases} \frac{64}{15}x^3 & \text{for} \; -0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases} \) a valid pdf?

No, pdfs cannot be negative.
2. (40%) A car salesman tries to sell two cars, a BMW and a Ford, that are assigned to him by his manager. Now consider the random experiment that a customer comes into the dealership. The customer may either buy one of the two cars or not buy any car at all. Assume that it is equally likely for the customer to buy or not buy a car. If the customer does buy a car, it is 4 times more likely that (s)he will choose the Ford than choosing the BMW.

(a) (12%) Specify the probability space that describes this random experiment. You have to list out all possible events and their corresponding probabilities.

(b) (8%) Consider that the salesman receives a commission of $1000 if the Ford is sold and $2000 if the BMW is sold. Let $X$ be the random variable that records the amount of commission that the salesman receives. Determine and draw the cdf of $X$.

(c) (6%) Now a second customer walks in and wants to buy a car. Since there are only two cars, if the first customer buys a car, then that car will not be available to the second customer. Determine the sample space that describes all the possible outcomes of the random experiment concerning the purchases of the two customers.

(d) (14%) Assume that if the first customer buys a car, then it is still equally like for the second customer to buy or not buy a car. On the other hand, if the first customer does not buy any car, then the second customer will choose in the same way as the first customer does, as described in the first paragraph of the problem. Now let $Y$ be the total commission that the salesman receives from the transactions of the two customers. Determine and draw the cdf of $Y$.

\[ \Omega = \{ B, F, N \} \]

\[ \mathcal{F} = \{ \emptyset, \Omega, \{ B \}, \{ F \}, \{ N \}, \{ B, F \}, \{ F, N \}, \{ B, N \} \} \]

\[ P(\emptyset) = 0, \quad P(\Omega) = 1, \quad P(\{N\}) = 0.5 \]

\[ P(\{B\}) = 0.1, \quad P(\{F\}) = 0.4, \quad P(\{B, F\}) = 0.5 \]

\[ P(\{F, N\}) = 0.9, \quad P(\{B, N\}) = 0.6 \]

\[ F_X(x) \]

\[ 1 \quad 0.9 \quad 0.5 \quad 0 \]

\[ 1000 \quad 2000 \quad \rightarrow X \]
2. (c) \[ \Omega = \{ NB, NF, NN, BN, BF, FN, FB \} \]

\[ P(NB) = 0.5 \times 0.1 = 0.05 \]
\[ P(NF) = 0.5 \times 0.4 = 0.2 \]
\[ P(NN) = 0.5 \times 0.5 = 0.25 \]
\[ P(BN) = 0.1 \times 0.5 = 0.05 \]
\[ P(BF) = 0.1 \times 0.5 = 0.05 \]
\[ P(FN) = 0.4 \times 0.5 = 0.2 \]
\[ P(FB) = 0.4 \times 0.5 = 0.2 \]

\[ Y(NB) = 2000 \]
\[ Y(NF) = 1800 \]
\[ Y(NN) = 0 \]
\[ Y(BN) = 2000 \]
\[ Y(BF) = 3000 \]
\[ Y(FN) = 1000 \]
\[ Y(FB) = 3500 \]
3. (40%) A radar system detects the presence of an airplane based on the strength of the signal received at its receiver. If the signal strength exceeds the threshold $\gamma$, then the radar system will announce that the airplane is present. Let the random variable $X$ denote the received signal strength. If there is an airplane, $X \sim \mathcal{N}(A, \sigma)$. If there is no airplane, $X \sim \mathcal{N}(0, \sigma)$. This means that $X$ is Gaussian, with different means, in both cases. Assume that $\gamma < A$.

(a) (10%) Find the probability of the event that the signal strength exceeds the threshold $\gamma$. Express your results in terms of $A$, $\gamma$, $\sigma$, and the $\Phi(\cdot)$ function (the cdf of a standard Gaussian r.v.).

(b) (10%) Find the a posteriori probability of the event that the airplane is absent conditioned on the event that the signal strength exceeds the threshold $\gamma$.

(c) (10%) Find the a posteriori probability of the event that the airplane is present conditioned on the event that the signal strength does not exceed the threshold $\gamma$.

(d) (10%) A reasonable choice of $\gamma$ is $A/2$. Simplify your results in (b) and (c) for this choice of $\gamma$. Your final results should depend only on the factor $\frac{A}{2\sigma}$, which is referred to as the signal-to-noise ratio (SNR). As a radar engineer what value (large or small) of SNR would you like to achieve. Why?

\[
P(X > \gamma) = P(X > \gamma | A\text{P}) P(A\text{P}) + P(X > \gamma | \text{A\text{B}}) P(\text{A\text{B}})
\]

\[
\frac{1}{2} \left(1 - \Phi \left(\frac{A-\gamma}{\sigma}\right)\right) + \frac{1}{2} \left(1 - \Phi \left(\frac{\gamma}{\sigma}\right)\right)
\]

(b) \quad P(\text{AB} | X > \gamma) = \frac{P(X > \gamma | \text{AB}) P(\text{AB})}{P(X > \gamma)} = \frac{1 - \Phi \left(\frac{\gamma-A}{\sigma}\right)}{1 - \Phi \left(\frac{\gamma}{\sigma}\right) + 1 - \Phi \left(\frac{\sigma}{\sigma}\right)}

(c) \quad P(A\text{P} | X \leq \gamma) = \frac{P(X \leq \gamma | A\text{P}) P(A\text{P})}{P(X \leq \gamma)} = \frac{\Phi \left(\frac{\gamma-A}{\sigma}\right)}{\Phi \left(\frac{\sigma-A}{\sigma}\right) + \Phi \left(\frac{\gamma}{\sigma}\right)}

(d) \quad \gamma = A/2 \quad \Rightarrow \quad P(\text{AB} | X > \gamma) = \frac{1 - \Phi \left(\frac{\gamma}{\sigma}\right)}{1 - \Phi \left(\frac{\gamma}{\sigma}\right) + 1 - \Phi \left(\frac{\gamma}{\sigma}\right)} = \frac{-\Phi \left(-\frac{A}{2\sigma}\right)}{\Phi \left(\frac{A}{2\sigma}\right) + \Phi \left(-\frac{A}{2\sigma}\right)}

Both events correspond to $\Phi \left(-\frac{A}{2\sigma}\right) + \Phi \left(\frac{A}{2\sigma}\right) \approx 1$.

\[ \quad \wedge \quad \text{large decisions, i.e. want } \frac{A}{2\sigma} \text{ big } \Rightarrow \text{probs small!} \]