EEL6503: Spread Spectrum and CDMA

Problem Set 1 (Fall 2001)

(due Class 4)

1. Suppose $\{\phi_n(t)\}_{n=1}^N$ is an orthonormal basis for the signal space spanned by a set $\mathcal{S}$ of square-integrable signal waveforms. We represent a waveform $s(t)$ in $\mathcal{S}$ by a N-dimensional vector $\mathbf{s} = [s_1, s_2, \ldots, s_N]^T$ whose n-th coordinate s_n, for $n = 1, 2, \ldots, N$, is given by the inner product of $s(t)$ and $\phi_n(t)$, i.e.,

$$s_n = (s, \phi_n) \triangleq \int_{-\infty}^{\infty} s(t)\phi_n(t)dt.$$

   (a) Show that we can uniquely determine the signal $s(t)$ from the vector $\mathbf{s}$ or vice versa.

   (b) For any $s_m(t), s_k(t) \in \mathcal{S}$, prove the following two identities:

$$\begin{aligned} (s_m, s_k) &\triangleq \int_{-\infty}^{\infty} s_m(t)s_k(t)dt = \mathbf{s}_m^T\mathbf{s}_k, \\ d^2(s_m, s_k) &\triangleq \int_{-\infty}^{\infty} [s_m(t) - s_k(t)]^2 dt = \|\mathbf{s}_m - \mathbf{s}_k\|^2, \end{aligned}$$

   where $\mathbf{s}_m$ and $\mathbf{s}_k$ are the vector representatives of $s_m(t)$ and $s_k(t)$, respectively.

   (c) For any square-integrable signal $r(t)$ (may not $\in \mathcal{S}$) and $s(t) \in \mathcal{S}$, show that

$$\int_{-\infty}^{\infty} r(t)s(t)dt = \mathbf{r}^T\mathbf{s},$$

   where $\mathbf{r} = [r_1, r_2, \ldots, r_N]^T$ and $r_n = \int_{-\infty}^{\infty} r(t)\phi_n(t)dt$ for $n = 1, 2, \ldots, N$.

2. *Definition:* Let $\mathbf{X}$ be a random sample of a finite-dimensional random vector with a probability density function (pdf) $f(\mathbf{x}|\theta)$ indexed by a parameter $\theta \in \Omega$, and $\mathbf{Y} = \varphi(\mathbf{X})$ be a statistic of $\mathbf{X}$ whose pdf is given by $g(\mathbf{y}|\theta)$. Then $\mathbf{Y}$ is a *sufficient statistic* for θ if and only if

$$f(\mathbf{x}|\theta) = h(\mathbf{x})g(\varphi(\mathbf{x})|\theta),$$

where $h(\mathbf{x})$ does not depend on θ for every fixed value of $\mathbf{x}$.

Based on this definition, show that $\mathbf{r}_N$ (defined in Section 1.1) is sufficient for determining which signal is being sent in the sense that $\mathbf{r}_N$ is a sufficient statistic for m, given the observation $\mathbf{r}_K = [r_1, r_2, \ldots, r_K]^T$ for each $K \geq N$.

3. Suppose $s(t) = 0$ for $t \notin [0, T]$ in Figure 1.3. Show that the correlator and the matched filter are equivalent in the sense that they give the same output when the received signal $r(t)$ is the input to them.

4. Consider a 9-QAM communication system. The 9 equally probable signals are given by, for $i = 0, 1, \ldots, 8$,

$$s_i(t) = \begin{cases} \sqrt{2A}(i_1 - 1)\cos(\omega_c t) + \sqrt{2A}(i_2 - 1)\sin(\omega_c t), & 0 \leq t < T, \\ 0, & \text{otherwise,} \end{cases}$$

where $\omega_c T = 2n\pi$ for some integer n and

$$\begin{aligned} i_1 &= \lfloor i/3 \rfloor = \text{ the quotient of i divided by 3,} \\ i_2 &= i - 3\lfloor i/3 \rfloor = \text{ the remainder of i divided by 3.} \end{aligned}$$

Note that $\lfloor x \rfloor$ represents the largest integer that is not larger than x. Assume that the channel is corrupted by AWGN with two-sided spectral density $N_0/2$, and that ML receiver is used.

(a) Find an orthonormal basis for the 9 signals described above. Draw the signal constellation based on your orthonormal representation.

(b) Find the union bound on the conditional probability of symbol error given $s_0(t)$ is sent. You can assume that only the signals that are closest to $s_0(t)$ contribute significantly in the union bound; that is, you can neglect those signals that are not the nearest neighbors of $s_0(t)$ in your union bound expression. Find the union bounds for the case when $s_1(t)$ is sent and the case when $s_4(t)$ is sent. Again, you only need to consider the nearest neighbors in the union bounds.

(c) Find the bound on the average symbol error probability by using the results in (b).

5. If $n(t)$ is a WSS process with zero mean and power spectral density $\Phi_n(\omega)$ satisfying the narrowband assumption, then $n(t) = n_x(t)\cos(\omega_c t) - n_y(t)\sin(\omega_c t)$, where $n_x(t)$ and $n_y(t)$ are zero-mean jointly WSS processes. By employing the stationarity of the random processes involved, show that

$$\begin{aligned} R_{n_x}(\tau) &= R_{n_y}(\tau), \\ R_{n_x n_y}(\tau) &= -R_{n_y n_x}(\tau), \\ R_n(\tau) &= R_{n_x}(\tau)\cos(\omega_c \tau) - R_{n_x n_y}(\tau)\sin(\omega_c \tau). \end{aligned}$$

6. Suppose $s_m(t)$ for $m = 0, 1, \ldots, M-1$ are bandpass signals with complex envelopes $\tilde{s}_m(t)$. Rederive the coherent ML receiver in Section 1.1 using the complex envelope representation. Note that if $\{\tilde{\phi}_n\}_{n=1}^{\infty}$ is an orthonormal basis for the space of complex square-integrable signals, then a complex square-integrable signal $\tilde{r}(t)$ can be represented by a complex vector $\tilde{\mathbf{r}} = [\tilde{r}_1, \tilde{r}_2, \ldots]^T$ whose coordinates are given by, for $k = 1, 2, \ldots$,

$$\tilde{r}_k = \int_{-\infty}^{\infty} \tilde{r}(t)\tilde{\phi}_k^*(t)dt.$$

Moreover, if the receiver vector $\tilde{\mathbf{r}} = \tilde{\mathbf{s}}_m + \tilde{\mathbf{n}}$, where the coordinates of $\tilde{\mathbf{n}}$ are iid complex symmetric Gaussian random variables with zero mean and variance N_0, then the conditional pdf

$$p(\tilde{r}_k|\tilde{s}_{mk}) = \frac{1}{2\pi N_0}\exp\left[-\frac{1}{2N_0}|\tilde{r}_k - \tilde{s}_{mk}|^2\right],$$

for $k = 1, 2, \ldots, N$, where N is the dimension of the signal space $\{\tilde{s}_m(t)\}_{m=0}^{M-1}$.