EEL6503 : Spread Spectrum and CDMA

< Solution of Problem Set 1 >

1. (a) We want to prove that there is an one-to-one correspondence between $s(t)$ and $\mathbf{s}$.

We will show that if $s_1(t) = s_2(t)$, $\mathbf{s}_1 = \mathbf{s}_2$ and if $\mathbf{s}_1 = \mathbf{s}_2$, $s_1(t) = s_2(t)$.

i) $s_{1n} = \int_{-\infty}^{\infty} s_1(t) f_n(t) dt$, $s_{2n} = \int_{-\infty}^{\infty} s_2(t) f_n(t) dt$

If $s_1(t) = s_2(t)$, then $s_{1n} = s_{2n}$ ➔ $\mathbf{s}_1 = \mathbf{s}_2$

ii) $s_1(t) = \sum_{n=1}^{N} s_{1n} f_n(t)$, $s_2(t) = \sum_{n=1}^{N} s_{2n} f_n(t)$

If $\mathbf{s}_1 = \mathbf{s}_2$ $\left(i.e., s_{1n} = s_{2n}, n = 1, 2, \cdots, N\right)$, then $s_1(t) = s_2(t)$

So, we can uniquely determine the signal $s(t)$ from the vector $\mathbf{s}$ or vice versa.

(b)

$$
\begin{aligned}
(s_m, s_k) &\triangleq \int_{-\infty}^{\infty} s_m(t) s_k(t) dt \\
&= \int_{-\infty}^{\infty} \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{n'=1}^{N} s_{kn'} f_{n'}(t) dt \\
&= \sum_{n=1}^{N} \sum_{n'=1}^{N} s_{mn} s_{kn'} \int_{-\infty}^{\infty} f_n(t) f_{n'}(t) dt = \sum_{n=1}^{N} \sum_{n'=1}^{N} s_{mn} s_{kn'} d_{n-n'} \\
&= \sum_{n=1}^{N} s_{mn} s_{kn'} = \mathbf{s}_m^T \mathbf{s}_k
\end{aligned}
$$

$$
\begin{aligned}
d^2(s_m, s_k) &\triangleq \int_{-\infty}^{\infty} [s_m(t) - s_k(t)]^2 dt \\
&= \int_{-\infty}^{\infty} \left[s_m^2(t) - 2 s_m(t) s_k(t) + s_k^2(t)\right] dt \\
&= \int_{-\infty}^{\infty} \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{n'=1}^{N} s_{mn'} f_{n'}(t) dt - 2 \int_{-\infty}^{\infty} \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{n'=1}^{N} s_{kn'} f_{n'}(t) dt \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{N} s_{kn} f_n(t) \sum_{n'=1}^{N} s_{kn'} f_{n'}(t) dt \\
&= \sum_{n=1}^{N} s_{mn}^2 - 2 \sum_{n=1}^{N} s_{mn} s_{kn} + \sum_{n=1}^{N} s_{kn}^2 \\
&= \left\| \mathbf{s}_m - \mathbf{s}_k \right\|^2
\end{aligned}
$$

(c) $r(t)$ may $\notin \mathbf{S}$

Define $r'(t)=\sum_{n=1}^{N} r_n f_n(t)$, Then $r'(t)\in \mathbf{S}$ and $r(t)-r'(t)\in \mathbf{S}_{\perp}$.

$$\int_{-\infty}^{\infty}[r(t)-r'(t)]f_n(t)dt$$
$$=\int_{-\infty}^{\infty} r(t)f_n(t)dt-\int_{-\infty}^{\infty} r'(t)f_n(t)dt$$
$$=r_k-r_k=0 \qquad \forall\ n=1,2,\cdots,N$$

So,

$$\int_{-\infty}^{\infty}[r(t)-r'(t)]s(t)dt$$
$$=\int_{-\infty}^{\infty}[r(t)-r'(t)]\sum_{n=1}^{N} s_n f_n(t)dt=0$$

Let $r(t)=[r(t)-r'(t)]+r'(t)$.

$$\int_{-\infty}^{\infty} r(t)s(t)dt=\int_{-\infty}^{\infty}\{[r(t)-r'(t)]+r'(t)\}s(t)dt$$
$$=\int_{-\infty}^{\infty}[r(t)-r'(t)]s(t)dt+\int_{-\infty}^{\infty} r'(t)s(t)dt$$
$$=\int_{-\infty}^{\infty} r'(t)s(t)dt$$
$$=\int_{-\infty}^{\infty}\sum_{n=1}^{N} r_n f_n(t)\sum_{n'=1}^{N} s_{n'} f_{n'}(t)dt=\sum_{n=1}^{N} r_n s_n$$
$$=\mathbf{r}^T\mathbf{s}$$

where $\qquad \mathbf{r}=[r_1,r_2,\cdots\cdots,r_N]^T$

Method 2:

$$\int_{-\infty}^{\infty} r(t)s(t)dt=\int_{-\infty}^{\infty} r(t)\sum_{n=1}^{N} s_n f_n(t)dt$$
$$=\sum_{n=1}^{N} s_n\int_{-\infty}^{\infty} r(t)f_n(t)dt$$
$$=\sum_{n=1}^{N} r_n s_n$$
$$=\mathbf{r}^T\mathbf{s}$$

2. $\quad \mathbf{r}=[r_1,r_2,\cdots\cdots,r_N,r_{N+1},\cdots\cdots,r_K]^T, \qquad N\le K$

$$\begin{cases} r_j=s_{mj}+n_j, & if\ 1\le j\le N \\ r_j=n_j, & if\ N+1\le j\le K \end{cases}$$

$$f\left(\mathbf{r}_K \middle| \mathbf{s}_m\right) = \underbrace{\prod_{j=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{n_j}^2}} \exp\left\{-\frac{\left(r_j - s_{mj}\right)^2}{2\sigma_{n_j}^2}\right\}}_{g\left(\phi\left(\mathbf{r}_K\right)|\mathbf{s}_m\right)=g\left(\mathbf{r}_N|\mathbf{s}_m\right)} \underbrace{\prod_{j=N+1}^{K} \frac{1}{\sqrt{2\pi\sigma_{n_j}^2}} \exp\left\{-\frac{r_j^{\,2}}{2\sigma_{n_j}^2}\right\}}_{h\left(\mathbf{r}_K\right)}$$

So, based on the definition of a sufficient statistic, $\mathbf{r}_N = \left[r_1, r_2, \cdots\cdots, r_N\right]^T$ is a sufficient statistic for m.

3. From the output of the matched filter followed by sampler,

$$\begin{aligned} & r(t) * s(T-t)\big|_{t=T} \\ & = \int_{-\infty}^{\infty} r(t)s\left(t-(T-t)\right)dt \bigg|_{t=T} \\ & = \int_{0}^{T} r(t)s(t)dt \end{aligned}$$

which is the same as the output of the correlator.

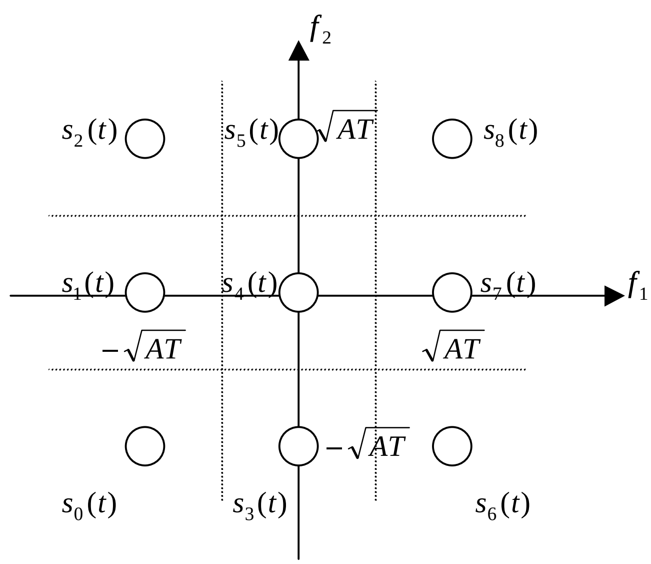
4. (a)

$$f_1(t) = \sqrt{\frac{2}{T}} \cos\left(\omega_c t\right) p_T(t)$$

$$f_2(t) = \sqrt{\frac{2}{T}} \sin\left(\omega_c t\right) p_T(t)$$

where

$$p_T(t) = \begin{cases} 1, & if\ 0 \le t \le T \\ 0, & otherwise \end{cases}$$



(b) $P_{s|0} \le Q\left(\frac{d(s_{02}, s_{12})}{2s_n}\right) + Q\left(\frac{d(s_{02}, s_{32})}{2s_n}\right) = 2Q\left(\sqrt{\frac{AT}{2N_0}}\right)$

$$P_{s|1} \le Q\left(\frac{d(s_{12}, s_{02})}{2s_n}\right) + Q\left(\frac{d(s_{12}, s_{22})}{2s_n}\right) + Q\left(\frac{d(s_{12}, s_{42})}{2s_n}\right) = 3Q\left(\sqrt{\frac{AT}{2N_0}}\right)$$

$$P_{s|4} \le 4Q\left(\sqrt{\frac{AT}{2N_0}}\right)$$

(c) $P_s \le \frac{1}{9}\left\{4 \times 2Q\left(\sqrt{\frac{AT}{2N_0}}\right) + 4 \times 3Q\left(\sqrt{\frac{AT}{2N_0}}\right) + 4Q\left(\sqrt{\frac{AT}{2N_0}}\right)\right\} = \frac{24}{9}Q\left(\sqrt{\frac{AT}{2N_0}}\right)$

5. $n(t) = n_x(t)\cos(w_c t) - n_y(t)\sin(w_c t)$

$$\begin{aligned}
R_n(t, t+t) &= E\{n(t)n(t+t)\} \\
&= E\{[n_x(t)\cos(w_c t) - n_y(t)\sin(w_c t)][n_x(t+t)\cos(w_c(t+t)) - n_y(t+t)\sin(w_c(t+t))]\} \\
&= E\{n_x(t)n_x(t+t)\}\cos(w_c t)\cos(w_c(t+t)) - E\{n_x(t)n_y(t+t)\}\cos(w_c t)\sin(w_c(t+t)) \\
&\quad - E\{n_y(t)n_x(t+t)\}\sin(w_c t)\cos(w_c(t+t)) + E\{n_y(t)n_y(t+t)\}\sin(w_c t)\sin(w_c(t+t)) \\
&= R_{n_x}(t)\cos(w_c t)\cos(w_c(t+t)) - R_{n_x n_y}(t)\cos(w_c t)\sin(w_c(t+t)) \\
&\quad - R_{n_y n_x}(t)\sin(w_c t)\cos(w_c(t+t)) + R_{n_y}(t)\sin(w_c t)\sin(w_c(t+t)) \\
&= R_{n_x}(t)\frac{1}{2}[\cos(w_c t) + \cos(w_c(t+t))] - R_{n_x n_y}(t)\frac{1}{2}[\sin(w_c t) + \cos(w_c(t+t))] \\
&\quad - R_{n_y n_x}(t)\frac{1}{2}[-\sin(w_c t) + \sin(w_c(t+t))] + R_{n_y}(t)\frac{1}{2}[\cos(w_c t) - \cos(w_c(t+t))] \\
&= \frac{1}{2}[R_{n_x}(t) + R_{n_y}(t)]\cos(w_c t) + \frac{1}{2}[R_{n_x}(t) - R_{n_y}(t)]\cos(w_c(2t+t)) \\
&\quad + \frac{1}{2}[R_{n_x n_y}(t) - R_{n_y n_x}(t)]\sin(w_c t) - \frac{1}{2}[R_{n_x n_y}(t) + R_{n_y n_x}(t)]\sin(w_c(2t+t))
\end{aligned}$$

Since $n(t)$ is a WSS process, the autocorrelation for $n(t)$ can be made to be a function of t only if the terms involving t are equal to be zero.

Then,
$$R_{n_x}(t) = R_{n_y}(t)$$

$$R_{n_x n_y}(t) = -R_{n_y n_x}(t)$$

and

$$R_n(t) = R_{n_x}(t)\cos(w_c t) - R_{n_x n_y}(t)\sin(w_c t)$$

6.

$$s_m(t) = \mathrm{Re}\left\{\tilde{s}_m(t)e^{jw_c t}\right\}$$

$$r(t) = \mathrm{Re}\left\{\tilde{r}(t)e^{jw_c t}\right\}$$

$$\tilde{r}_k = \int_{-\infty}^{\infty} \tilde{r}(t)\tilde{f}_k^*(t)dt$$

$$p(\tilde{r}_k|\tilde{s}_{mk}) = \frac{1}{2pN_0}\exp\left[-\frac{|\tilde{r}_k - \tilde{s}_{mk}|^2}{2N_0}\right] \quad \text{for } k = 1, 2, \cdots\cdots, N$$

Let $\tilde{\mathbf{r}}_N = [\tilde{r}_1, \tilde{r}_2, \cdots\cdots, \tilde{r}_N]^T$

$$p(\tilde{\mathbf{r}}_N|\tilde{\mathbf{s}}_{mN}) = \prod_{k=1}^{N}\frac{1}{2pN_0}\exp\left[-\frac{|\tilde{r}_k - \tilde{s}_{mk}|^2}{2N_0}\right]$$

Since logarithm is a monotonic increasing function, by taking logarithm of $p(\tilde{\mathbf{r}}_N|\tilde{\mathbf{s}}_{mN})$ it is easy to see that the ML receiver picks $m \in \{0, 1, \cdots\cdots, M-1\}$ such that the squared Euclidean distance between the signal vector $\tilde{\mathbf{s}}_{mN}$ and the vector $\tilde{\mathbf{r}}_N$,

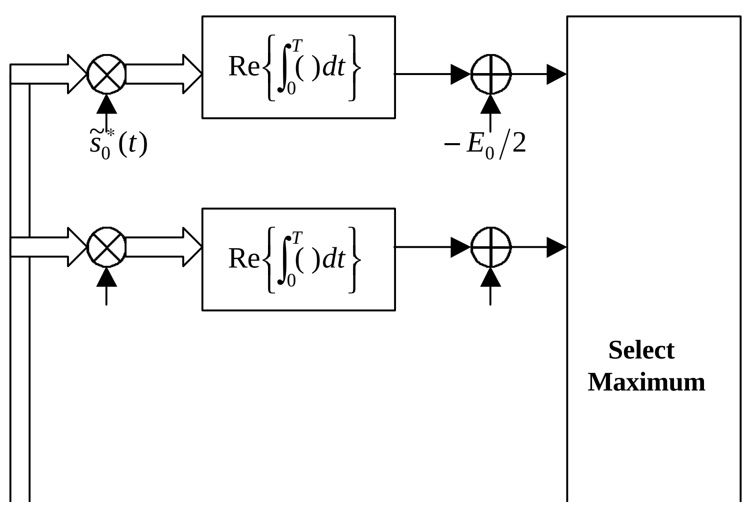
$$d^2(\tilde{\mathbf{s}}_{mN}, \tilde{\mathbf{r}}_N) = \sum_{k=1}^{N}(\tilde{s}_{mk} - \tilde{r}_k)^2 = \tilde{\mathbf{s}}_{mN}^T\tilde{\mathbf{s}}_{mN}^* - 2\,\mathrm{Re}\left\{\tilde{\mathbf{s}}_{mN}^{*\,T}\tilde{\mathbf{r}}_N\right\} + \tilde{\mathbf{r}}_N^T\tilde{\mathbf{r}}_N^*$$

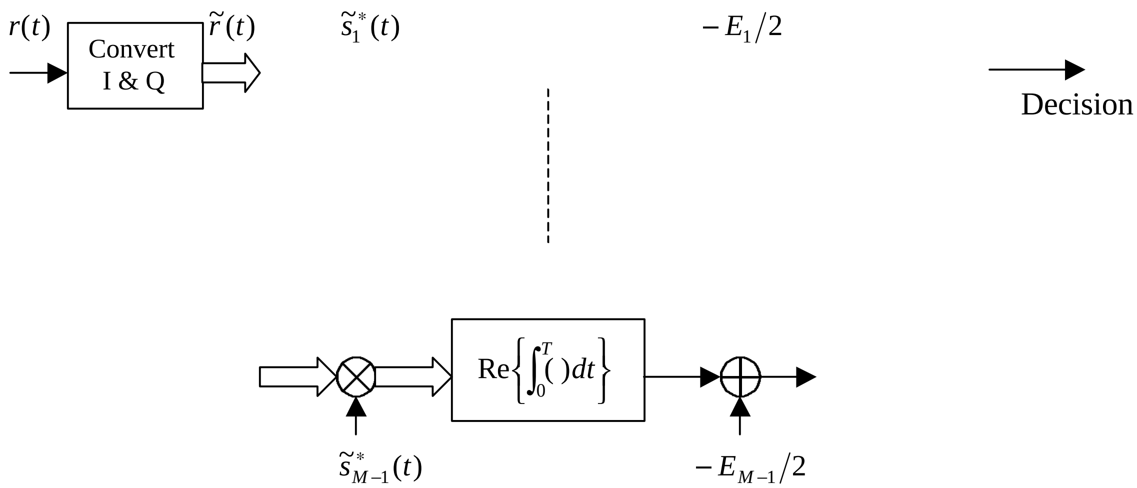
is minimized.

$$\arg\max_{m\in\{0,1,\cdots,M-1\}} d^2(\tilde{\mathbf{s}}_{mN}, \tilde{\mathbf{r}}_N) = \arg\max_{m\in\{0,1,\cdots,M-1\}} C(\tilde{\mathbf{s}}_{mN}, \tilde{\mathbf{r}}_N)$$

where

$$C(\tilde{\mathbf{s}}_{mN}, \tilde{\mathbf{r}}_N) = \mathrm{Re}\left\{\tilde{\mathbf{s}}_{mN}^{*\,T}\tilde{\mathbf{r}}_N\right\} - \frac{1}{2}\tilde{\mathbf{s}}_{mN}^{\ T}\mathbf{s}_{mN}^*$$
$$= \mathrm{Re}\left\{\int_0^T \tilde{r}(t)\tilde{s}_m^*(t)dt\right\} - \frac{1}{2}E_m$$





< coherent ML receiver using the complex envelope representation >