**Conditional Probability, Bayes’ Rule, Maximum A Posteriori Decision Rules**

**SS-1.** Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the PSA (prostate specific antigen) protein that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately 0.135, with this probability increasing to approximately 0.268 if the man does have cancer. If, based on other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that

(a) the test indicated an elevated PSA level? *Ans:* 0.822

(b) the test did not indicate an elevated PSA level? *Ans:* 0.6638

Repeat the preceding, this time assuming that the physician initially believes that there is a 30 percent chance that the man has prostate cancer. Answers: If PSA elevated: 0.46, if not elevated: 0.266.

**SS-2.** A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, or weak are 0.7, 0.2, and 0.1, respectively.

(a) How certain is she that she will receive the new job offer? (*Ans:* \(P(\text{gets job offer})=0.65\))

(b) Given that she does receive the job offer, how likely should she feel that she received a strong recommendation (*Ans:* 56/65); a moderate recommendation(*Ans:* 8/65); a weak recommendation (*Ans:* 1/65)?

(c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation (*Ans:* 14/35); a moderate recommendation(*Ans:* 12/35); a weak recommendation (*Ans:* 9/35)?

1. Consider the binary communication system shown below.

Let \(p_{01} < 1/2\) and \(p_{10} < 1/2\).

(a) For what values of \(p_0 = P(A_0)\) are the MAP and ML decision rules different? Give your answer in terms of the channel transition probabilities \(p_{01}\) and \(p_{10}\).

(b) Find all the possible MAP decision rules when \(p_{01} = 1/8\) and \(p_{10} = 1/6\). Specify the range of \(p_0\) for which each MAP decision rule holds.
2. From *Random Signal Analysis in Engineering Systems* by John J. Komo (slightly modified)
For the digital communication system shown below, where $P(A_0) = 0.6$ and $P(A_1) = 0.4$, completely specify the MAP decision rule and calculate the overall probability of error under that rule.

SS-3. You are sent as part of a UN inspection team to investigate whether North Korea has weapons-grade plutonium (WGP) in a reactor. Let $W$ denote the event that WGP is present.
You take measurements with a Geiger counter and decide that WGP is detected if some threshold is exceeded. Suppose that the threshold is set so that the probability of correct detection, $P(D|W)$, is 0.9, and the probability of false detection, $P(D|\overline{W})$ is 0.25.
Suppose that the probability that WGP is present is 0.3.

*It may help to draw a diagram showing the transition probabilities.*

(a) What is the probability that WGP is detected? (0.445) Not detected? (0.555)
(b) If you detect WGP, what is the probability that it is actually present? (0.607)
(c) If you detect WGP, what is the probability that it is not present? (The conditional probability of false alarm). (0.393)
(d) If you do not detect WGP, what is the probability that it was actually there? (The conditional probability of a miss). (0.054)
(e) What is the overall probability that you make the wrong conclusion based on your measurements? (0.205)
3. On a very noisy communication channel, the probability of a packet being received correctly is 0.2. If the packet is not received correctly, then the destination does not send an acknowledgment (ACK). The source repeatedly transmits the packet until an ACK is received.

(a) What is the probability that three transmissions are required?
(b) What is the probability that more than three transmissions are required?
(c) Given that the first three transmissions were failures, what is the probability that more than three more transmissions are required?
(d) Find a formula for the probability that $\Delta$ more transmissions are required given that the first $j$ transmissions were failures.
(e) Explain why geometric probabilities are said to be “memoryless”.

4. In 2003, there were many media reports about the number of shark attacks in Florida. At the end of the year, there were a total of 30 unprovoked shark attacks. By comparison, there were 246 shark attacks over the prior ten years.

(a) Give an expression for the probability of more than thirty shark attacks occurring in a year based on the historical data (don’t include the data for 2003). Use a Gaussian approximation to give an approximate numerical answer.
(b) Suppose that the probability that there are more than 30 shark attacks in a given year is 0.1. Find the probability that there is at least one year with more than 30 shark attacks in 10 year period.
(c) Use the historical data to answer this question. Shark attacks occur primarily during warm weather, so suppose that shark attacks only occur over a 30 week period. In 2003, the media made a big deal about two shark attacks during the same week. What is the probability of having at least two shark attacks during a given week?
(d) Using the assumptions of the previous part of this problem, what is the probability of having at least one week with three or more shark attacks during a given year?

5. Do problem 1.39 in the textbook, adding the following parts to it:

(c) Show that the probability that you found in part (a) is a binomial probability. In other words, show that the probability can be written as

$$\binom{N}{n_1} p^{n_1}(1 - p)^{N-n_1}.$$  

Express $N$ and $p$ as functions of $n_1, n_2, t_1$ and $t_2$.

(d) Give an intuitive explanation for the formula for $p$ in part (c).
SS-4. A dart is thrown into a square with corners at \((0, 0), (0, b), (b, 0)\) and \((b, b)\). Assume that the dart is equally likely to fall anywhere in the square. Let the random variable \(Z\) be given by the sum of the two coordinates of the point where the dart lands.

(a) Describe the sample space of \(Z\). (Ans: \(\Omega_z = \{z|0 \leq z \leq 2b\}\))

(b) Find the region in the square corresponding to the event \(\{Z \leq z\}\) for \(-\infty < z < \infty\).

(c) Find and plot \(F_Z(z)\)

\[F_Z(z) = \begin{cases} 
0, & z \leq 0 \\
\frac{z^2}{2b^2}, & 0 < z \leq b \\
\frac{1}{b^2} - \frac{(2b-z)^2}{b^2}, & b < z < 2b \\
1, & z \geq 2b
\end{cases}\]

(d) Specify the type (discrete, continuous, mixed) of \(Y\). (continuous)

6. The sample space for a random experiment is shown as the shaded area in the figure below. All outcomes in \(S\) are equally likely. A random variable \(Y\) is set equal to the \(y\)-coordinate of an outcome \(s = (x, y)\).

(a) Find the distribution function of \(Y\).

(b) Determine the type (discrete, continuous, mixed) of the random variable \(Y\).

SS-5. A random variable \(X\) has the distribution function shown in the figure at the top of the next page.

(a) What type of random variable is \(X\)?
(b) Find the density function for $X$.

(c) Give an equation for the probability $P(0.5 < X \leq 2)$ in terms of the distribution function for $X$.

(d) Give an equation for the probability $P(0.5 < X \leq 2)$ in terms of the density function for $X$.

(e) Give the numerical value of the probability $P(1 < X \leq 2)$.

(f) Give the numerical value of the probability $P(1 \leq X \leq 2)$.

7. A random variable has the density function shown below, where $c$ is a constant.

(a) Find $f_X(x)$.

(b) Find $F_X(x)$.

(c) Find $b$ such that $P(|X| < b) = 1/2$.

8. Let $X$ be a binomial random variable that results from the performance of $n$ Bernoulli trials with probability of success $p$.

(a) Suppose that $X = 1$. Find the probability that the single event occurred in the $k$th Bernoulli trial.
(b) Suppose that $X = 2$. Find the probability that the two events occurred in the $j$th and $k$th Bernoulli trials, where $j < k$.

(c) In light of your answers to parts a and b, in what sense are successes distributed “completely at random” over the $n$ Bernoulli trials?

SS-7. Let $X$ be a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. Find the following probabilities

(a) $P[X < \mu]$ (Ans: 1/2)

(b) $P[|X - \mu| > k\sigma]$ for $k = 1, 2, 3, 4, 5$ (Ans: 0.317, $4.55 \times 10^{-2}$, $2.70 \times 10^{-3}$, $6.34 \times 10^{-5}$, and $5.75 \times 10^{-7}$, respectively)