Homework 9  
Due Monday, December 3, 2007

Read pp. 552-564 in the textbook and do the following problems:

1. Let $X$ and $Y$ be random variables with joint density function

$$f_{XY}(x, y) = x + y, \quad 0 < x < 1, 0 < y < 1$$

(a) Find the minimum mean-square error linear estimator for $Y$ given $X$
(b) Find the mean-square error for the maximum-likelihood estimator for $Y$ given $X$ (the estimator that, when observing $X = x$, picks the estimate $y$ as the value that maximizes $f_{Y|X}(y|x)$).
(c) Compare both of the above estimators to the best minimum mean-square error estimator by calculating the mean-square error for each estimator and then drawing conclusions.

2. Let $Y = X + N$, where $X$ and $N$ are independent zero-mean Gaussian random variables with different variances.

(a) Plot the correlation coefficient between the “observed signal” $Y$ and the “desired signal” $X$ as a function of the signal-to-noise ratio $\sigma_X/\sigma_N$.
(b) Find the minimum mean-square error estimator for $X$ in terms of $Y$. Find the mean-square error for the estimator.

Read Sections 6.1 and 7.1 in the textbook and answer the following questions:

SS-1. Determine the classification of the random processes in each of the following situations. In each case, indicate if it is a continuous- or discrete-time process and if it is a continuous- or discrete- amplitude process.

(a) A manufacturing process begins at time 0, and we are interested in the number of defects that have occurred up to time $t$ for all positive values of $t$.
(b) A computation is carried out in a sequence of steps in a special-purpose digital computer, and the content of a particular shift-register is converted to decimal form and recorded at the end of each clock cycle.
(c) A continuous-time signal is sampled and quantized every $T_0$ seconds, and we are interested in the quantized value of the signal at each sampling time.
(d) Because of man-made and thermal noise in an analog FM receiver, the demodulated audio signal differs from the transmitted audio signal. We are interested in the error signal (the difference between the transmitted and demodulated audio signal).

3. Suppose $X$ is a random variable that is uniformly distributed on $[0, 1]$. The random process $Y(t), t > 0$ is defined by $Y(t) = \exp\{-Xt\}$. Find the one-dimensional distribution function $F_{Y_1}(y; t)$ for the random process $Y(t)$. Find the one-dimensional density function $f_{Y_1}(y; t)$. 
4. Consider the continuous-amplitude, continuous-time random process \( X(t) \), \( t \in \mathbb{R} \), which is defined by
\[
X(t) = Y_1 + tY_2,
\]
where \( Y_1 \) and \( Y_2 \) are independent Gaussian random variables, each having zero mean and variance \( \sigma^2 \). Find the one- and two-dimensional density functions for this random process.

*Hint:* \( X(t_1) \) and \( X(t_2) \) are jointly Gaussian, and may be correlated. The means, variances, and correlation coefficient is sufficient to specify the bivariate Gaussian density.

5. (A zero-mean Gaussian random process has autocorrelation function
\[
R_X(\tau) = \frac{1}{4 + \tau^2}, \quad \tau \in \mathbb{R}.
\]

The process is sampled at times \( t = 0, 1, \) and \( 2 \) to form the random vector \( \mathbf{Y} = [X(0) \ X(1) \ X(2)]^T \).

a) Find the density function of \( \mathbf{Y} \).

b) Let \( \mathbf{Y} = [1 \ 1/2 \ 1/4] \mathbf{Y} \), and find \( P(Y > 4) \). Give your answer as a number.

c) Find a linear transformation \( \mathbf{Z} = \mathbf{A} \mathbf{Y} \) such that \( \mathbf{Z} \) is a vector of iid Gaussian RVs with variance 1.
d) Translate problem (b), i.e., $P(Y > 4)$, into an equivalent problem on $Z$, $P(b Z > 4)$ and solve for the answer. (I.e. find vector $b$ that makes this prob. = $P(E_1 1/4) Y > 4$.) Show that $P(b Z > 4)$ is easier to solve.