3. **Convolution & Linear Filtering**

Previously, we consider connecting the Tx and Rx directly and the channel only adds noise to the transmitted signal (Actually, the noise is due to the jittering electronics in the Tx and Rx circuits). Of course, you can imagine that connecting the Tx and Rx directly is not at all realistic. Why do we need communication if the Tx and Rx are connected directly?

**Q:** So how can we describe the effect of a "real" channel, say, the telephone line or a wireless channel?

**A:** Well, there is a very difficult question to answer. It is, in fact, the objective of this course to give some simple and elementary answers to the question. Again, we start with the simplest case. Assuming that the channel is not time-varying and is relatively "well-behaved," we can usually model it as a **Linear Time-Invariant (LTI)** system.

The telephone line is a good example of this type of channels.
Q: OK, then what is a LTI system?

A: First, recall that a system takes in an input signal and splits out an output signal.

\[ x(t) \rightarrow T \rightarrow y(t) \]

The usual way to write the output as a result of a system acting on the input is as follows:

Suppose that \( T \) is a system, then

\[ y(t) = T[x(t)] \]

Def: By a LTI system, we mean that

(i) \[ y(t) = T[ax_1(t) + a_2x_2(t)] = a_1T[x_1(t)] + a_2T[x_2(t)] \]

(ii) if \( y(t) = T[x_1(t)] \), then \( y(t-t_0) = T[x_1(t-t_0)] \)

for all signals \( x_1(t) \) and \( x_2(t) \) and all numbers \( a_1, a_2, t_0 \).

This is to say, we get exactly the same output from the LTI system by doing either one of the following 2 things:

(i) Input scaled versions of 2 signals to the system,

(ii) Input each signal individually into the system and then take the sum of the scaled versions of the 2 output signals.

Also, if we delay the input to the LTI system, the output is simply a delayed version of the original output. The system does change in time!
Q: The definition of a LTI system is pretty cool, but it still doesn't say much about the system. How can we further characterize a specific LTI system?

A: We can use something called the impulse response of a LTI system to completely characterize a LTI system. To understand impulse response, we have to first introduce something called the impulse function (or the Dirac-delta function) $\delta(t)$.

Q: What is the impulse function?

A: Impulse function $\delta(t)$ is a weird object. We call it a "function", but it is actually not a function, nevertheless we will treat it as if it was a function.

(Of course, we will ignore all the math details.)

For us, the delta function $\delta(t)$ has the following properties:

(i) $\delta(t) = 0$ for all $t \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$

(iii) $\int_{-\infty}^{\infty} x(t) \delta(t) \, dt = x(0)$ if $x(t)$ is continuous at $t = 0$

($\text{or } \int_{-\infty}^{\infty} x(t) \delta(s-t) \, dt = x(s)$ if $x(t)$ is continuous at $t = s$)
Q: All right, so what is the impulse response of a LTI system?

A: As its name implies, the impulse response is the output of the LTI system when the impulse function is the input. Using our notation before, for the LTI system $T$, its impulse response is

$$R(t) = T[S(t)]$$

To find the impulse response of a LTI system, we input an impulse to the system and measure the output. (This may not be practical for some channels.)

Q: Having found the impulse response of a LTI system, how exactly can we characterize the system by the impulse response?

A: If for every input signal $x(t)$ we can tell what the output will be, then effectively we have characterized a system. We will show that this can be done using the impulse response of a LTI system.
Assume that the input signal \( x(t) \) is continuous (a reasonable assumption for practical signals). Then the output \( y(t) \) would be

\[
y(t) = T[x(t)] \\
= T \left[ \int_{-\infty}^{\infty} x(u) s(t-u) \, du \right] \quad \text{(prop iv) of } s(t) \\
= \int_{-\infty}^{\infty} x(u) T[s(t-u)] \, du \quad \text{(Linearity of } T) \\
= \int_{-\infty}^{\infty} x(u) h(t-u) \, du \quad \text{(Time-invariance of } T)
\]

Therefore, for an input \( x(t) \) to the LTI system, we can obtain the output \( y(t) \) by performing the above integral which only involves the input \( x(t) \) and the impulse response of the system.

Hence knowing the impulse response of the system is the same as knowing the system itself.

\[
Q: \text{How do people refer to the special integral above?} \\
A: \text{The integral is usually referred to as a convolution integral. In EE, we usually say it is the convolution between } x(t) \text{ and } h(t) \text{ (or } x(t) \text{ convolves with } h(t) \text{ or vice versa). Also we'd like to use the notation} \\
y(t) = x \star h(t) = \int_{-\infty}^{\infty} x(u) h(t-u) \, du \\
\text{(sometimes } x(t) \star h(t)\text{)}
\]
Q: What kinds of properties does the convolution operator have?

A: (i) Convolution with delta function:
\[ x(t) * \delta(t) = x(t), \]
\[ x(t) * \delta(t-t_0) = x(t-t_0). \]

(Pf: prop (iii) of delta function)

(ii) Commutativity:
\[ x(t) * y(t) = y(t) * x(t). \]

(Pf: \[ x(t) * y(t) = \int_{-\infty}^{\infty} x(u) y(t-u) \, du = \int_{-\infty}^{\infty} y(v) x(t-v) \, dv = y(t) * x(t). \])

(iii) Linearity:
\[ a_1 x_1(t) + a_2 x_2(t) * y(t) = a_1 [x_1 * y(t)] + a_2 [x_2 * y(t)]. \]

(Pf: This is just a fact reflecting the output of a LTI system can be represented by the convolution between the input and the impulse response of the system.)

Q: What are good about these properties?

A: I hope you recognize, by now, that we have to compute the convolution between the input and the impulse response of a LTI system in order to find the output. The properties above help us to break down a complicated convolution to simpler ones. You'll see this yourself in the examples below.
Q: The properties look simple but the convolution integral still look intimidating. Is there any way which helps to explain the convolution operator?

A: You bet. If you look carefully at the integral, you'll realize that it actually does the following steps:

1. Take one of the signals to be convolved and flip it horizontally about the vertical axis.

   You can pick either one of the two signals to flip since convolution is commutative.

   Ha, you've used the commutative prop. of convolution.

   So don't say the props are useless.

2. Put the flipped signal under the other signal and slide it to the right for positive t and to the left for negative t.

3. For each slide position (each t), multiply the two signals and calculate the area of the product. This is the value of the convolution for the particular t!

Easy enough? This is a good way to visualize the convolution operation.
Q: How about some examples?

A: Here you go:

**Eq. 1)** \( p(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \)

\[ p(t) \ast s(t) = ? \]

Two ways to do this:

(i) Use prop (i) of conv. We have \( p(t) \ast s(t) = p(t) \).

(ii) Do it pictorially. We choose to flip \( s(t) \).

\[ \begin{array}{c}
\vdots \\
T \\
\vdots \\
\end{array} \]

**Eq. 2)** \( p(t) \ast s(t-t_0) = ? \)

(i) Again by prop (i) of conv. \( p(t) \ast s(t-t_0) = p(t-t_0) \).

(ii) Pictorially, we choose to flip \( s(t-t_0) \).

Assume \( t_0 > 0 \)

\[ \begin{array}{c}
\vdots \\
T \\
\vdots \\
\end{array} \]

\[ t_0 \]

D: Do the \( t_0 < 0 \) case yourself!
Eq. 3) \[ p_1(t) \ast p_1(t) = ? \]

(i) You can evaluate the integral or

(ii) Pictorially,

\[ p_1(t) \]

\[ \rightarrow t \]

\[ \begin{cases} \frac{t}{2} , & 0 \leq t < T \\ 2-t , & T \leq t < 2T \\ 0 , & \text{otherwise} \end{cases} \]

Eq. 4) \[ p_1(t) \ast p_1(t) \ast p_1(t) = ? \]

We know \[ p_1(t) \ast p_1(t) = \]

So \[ p_1(t) \ast p_1(t) \ast p_1(t) = \]

\[ p_1(t) \]

\[ \begin{cases} \frac{t^3}{6} & , 0 \leq t < T \\ T^2 - \left(2Tt - \frac{(t-T)^2}{2}\right) & , T \leq t < 2T \\ \left(3T - T^2\right) & , 2T \leq t < 3T \\ 0 & , \text{otherwise} \end{cases} \]
\[ Eq. 5 \]

\[ + \]

\[ \begin{align*}
\Pr(t) \ast \Pr(t) & = \Pr \ast \Pr \ast \Pr(t) + \Pr \ast \Pr(t) \\
& + \Pr(t)
\end{align*} \]

We've used the linearity property of conv.

Q: Can we do convolution in MATLAB?

A: Of course. However, you have to remember that MATLAB works on digital computers and the signals are analog. Therefore, we have to first sample the signals and then approximate the convolution integral by a summation.

Suppose that we want to calculate \( Z(t) = \mathbf{x}(t) \ast \mathbf{y}(t) \). First sample the signals with sampling period \( T_s \),

\[ Z_k = Z(kT_s), \quad x_k = x(kT_s), \quad y_k = y(kT_s) \]

(Of course, put the samples into vectors in MATLAB.)

Then

\[ Z_k = \sum_{k=-\infty}^{\infty} x_k y_{k} \ast y_{k} \]

\[ \approx \sum_{k=-\infty}^{\infty} x_k y_{k} \ast y_{k} \]

\[ Z_k = \sum_{k=-\infty}^{\infty} x_k y_{k} \ast y_{k} \]  

(again, it is a usual practice to set \( T_s = 1 \))
From the results on the previous page, we know that the samples of $Z(t)$ are approximately given by a sum involving the samples of $X(t)$ & $Y(t)$. The sum is called a convolution sum. This is exactly the way we calculate convolution in MATLAB. You can either write a function to calculate the convolution sum or use the built-in function "conv" in MATLAB to do the job.

Q: Help conv in MATLAB!

Q: So far so good with all this math. But how do all these concepts come together to model a comm. channel?

A: First, recall that a large number of comm. channels can be modeled by LTI systems. In comm. terms, we usually say the comm. channels are LTI filters (or just say filters sometimes). To model a channel and use the model later on, we find the impulse response of the channel (filter). This can be done by sending an impulse to the channel and measuring the output. From the results involving convolution above, we then know that the effect of the channel is simply that the channel output will be the convolution between the input and the impulse response.
Moreover, we have to add thermal noise to the output too since thermal noise is basically due to the jittering electrons in the Rx circuit which has nothing to do with the channel itself.

Conclusion: Received signal seen by Rx is the sum of the convolution between the Tx signal and the channel impulse response and the thermal noise.

Pictorially,

$$ S(t) \rightarrow [R(t)] \rightarrow \ast \rightarrow h(t) \rightarrow \text{received signal} $$

**Tlx signal**

**Channel**

**Thermal noise**

(Special case: When Tx & Rx are connected directly, \( R(t) = S(t) \))

Mathematically,

\[ r(t) = s \ast R(t) + n(t) \]

Bottomline: Impulse response characterizes channel.